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Perturbation Analysis of a Stationary Nonequilibrium Flow Generated by an External Force

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The stationary flow of a gas in a slab under the action of a constant external force parallel to the walls is analyzed in the context of the Bhatnagar–Gross–Krook model kinetic equation. The force produces spatial gradients along the coordinate normal to the walls. By performing a perturbation expansion in powers of the force, we obtain the hydrodynamic fields up to fifth order in the force. Then the velocity distribution function and all its moments are evaluated to third order. The expansion coefficients are polynomials in the space variable of a degree increasing linearly with the expansion order. Although the series expansion is only asymptotic, it shows how the state of the system is modified by a variation of the external force beyond the linear regime.

KEY WORDS: Nonequilibrium steady state; Bhatnagar–Gross–Krook kinetic equation; external force; perturbation expansion.

1. INTRODUCTION

Nonequilibrium steady states are usually obtained in the laboratory by application of appropriate boundary conditions. From a theoretical, as well as a computer simulation, point of view, it is also useful to consider steady states driven out of equilibrium by the action of external forces. In the limit of small forces, the response of the system, as measured by the presence of hydrodynamic gradients and fluxes, is linear. In general, the strength of the forces provides a parameter measuring the departure of the system from equilibrium. Some examples of nonequilibrium steady states generated by external forces are the homogeneous heat conductivity(1,2) and color conductivity(3,4) problems.

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In this paper, we consider a gas enclosed between two infinite parallel plates at rest and kept at the same temperature. A constant external force **F** is applied along a direction, say x, parallel to the plates. In the steady state, the force **F** produces gradients of the hydrodynamic variables along the direction, say y, orthogonal to the plates. This flow has been analyzed by computer simulation of a lattice gas cellular automaton(5) and also theoretically in a dilute gas.(6,7)

At first order in the force, the resulting flow is equivalent to the well-known Poiseuille flow to Navier–Stokes order generated by a pressure gradient along the x direction.(6) Beyond that order, such an equivalence no longer holds and the steady flow driven by the external force is worth studying by itself. In ref. 6 an *exact* solution of the Bhatnagar–Gross–Krook (BGK) model kinetic equation was found for a *particular* value of the external force. The solution is characterized by a constant pressure and parabolic profiles of the flow velocity and the temperature with respect to an adequate space variable. That special value of the force is quite large, so that a physical picture of the nonlinear regime for moderate forces is still lacking.

The aim of this paper is to obtain rigorously the first few terms of the expansion in powers of the force of the general solution of the BGK equation for this problem. The results give evidence that the expansion is only asymptotic. For small forces, however, the expansion is useful for obtaining the hydrodynamic profiles, as well as the momentum and energy fluxes, beyond the linear domain. The organization of the paper is as follows. The BGK kinetic model and some basic definitions are presented in Section 2. In Section 3 the profiles of pressure, velocity, and temperature are obtained up to fifth order in the external force. The velocity distribution function and all its moments are derived in Section 4 up to third order in the force. Finally, the results are discussed in Section 5.

2. KINETIC MODEL

Let us assume that a dilute gas is found in a general nonequilibrium state in the presence of an external force **F**. The dilution condition allows us to use a kinetic theory description, according to which the state of the system is completely determined by the one-particle velocity distribution function $f(\mathbf{r}, \mathbf{v}; t)$. For a dilute gas, this function obeys the Boltzmann equation.(8) Nevertheless, the intricate structure of the Boltzmann collision term makes it difficult to obtain explicit solutions far from equilibrium. As a consequence, we will adopt the Bhatnagar–Gross–Krook (BGK) kinetic equation as a model of the Boltzmann equation.[8] The BGK equation reads

$$\frac{\partial}{\partial t} f + \mathbf{v} \cdot \nabla f + \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}}{m} f = -\nu(f - f_{\mathrm{LE}}) \tag{1}$$

The general idea behind Eq. (1) is the assumption that the net effect of collisions is to make f tend to relax with a characteristic frequency $\nu(\mathbf{r}, t)$ toward the local equilibrium distribution function

$$f_{\mathrm{LE}}(\mathbf{r}, \mathbf{v}; t) = n(\mathbf{r}, t) \left[\frac{m}{2\pi k_{\mathrm{B}} T(\mathbf{r}, t)} \right]^{3/2} \times \exp\left\{ -\frac{m}{2k_{\mathrm{B}} T(\mathbf{r}, t)} [\mathbf{v} - \mathbf{u}(\mathbf{r}, t)]^2 \right\} \tag{2}$$

In this equation, m is the mass of a particle, k_{B} is the Boltzmann constant, $n(\mathbf{r}, t)$ is the local number density, $\mathbf{u}(\mathbf{r}, t)$ is the local flow velocity, and $T(\mathbf{r}, t)$ is the local temperature. They are defined in terms of f as

$$n = \int d\mathbf{v}\, f \tag{3}$$

$$n\mathbf{u} = \int d\mathbf{v}\, \mathbf{v} f \tag{4}$$

$$n k_{\mathrm{B}} T = \frac{m}{3} \int d\mathbf{v}\, V^2 f \tag{5}$$

where $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}$ is the peculiar velocity. The five quantities n, $\mathbf{u}$, and T are the hydrodynamic fields and are associated with the densities of the conserved quantities (mass, momentum, and energy). The flux of mass is $mn\mathbf{u}$. The fluxes of momentum and energy are, respectively, the pressure tensor

$$P_{ij} = m \int d\mathbf{v}\, V_i V_j f \tag{6}$$

and the heat flux vector

$$\mathbf{q} = \frac{m}{2} \int d\mathbf{v}\, V^2 \mathbf{V} f \tag{7}$$

The trace of the pressure tensor is $3p$, where $p = n k_{\mathrm{B}} T$ is the hydrostatic pressure. The collision frequency ν scales with the density n. On the other hand, the interaction potential governing the collisions among the particles is introduced through the dependence of ν on T. For instance, $\nu \propto nT^{1/2}$ for hard spheres and $\nu \propto n$ for Maxwell molecules.

Equation (1) can be viewed as a nonhomogeneous linear equation for f. Its solution can be formally written as

$$f=(1+\mathscr{L})f_{\mathrm{LE}} \tag{8}$$

where $\mathscr{L}$ is a linear operator that incorporates the initial and boundary conditions, if necessary. Equation (8) gives a formal solution only, since f_{LE} is still a *nonlinear* functional of f through n, $\mathbf{u}$, and T. On the other hand, insertion of Eq. (8) into the right-hand sides of Eqs. (3)–(5) gives a closed set of five nonlinear coupled equations. Once its solution is known, Eq. (8) gives the explicit solution of the BGK equation for the physical problem of interest. Of course, this program is very difficult to achieve in practice for general nonequilibrium states. It has been recently carried out, for instance, for the steady Fourier and Couette flows, both numerically$^{(9)}$ (finite Knudsen number) and analytically$^{(10)}$ (zero Knudsen number).

Let us assume now that the gas is subject to the stationary flow described in Section 1. In that case, Eq. (1) becomes

$$v_y\frac{\partial f}{\partial y}+\frac{F}{m}\frac{\partial f}{\partial v_x}=-\nu(f-f_{\mathrm{LE}}) \tag{9}$$

We choose the plane $y=0$ as the one at equal distances from both plates, which are located at $y=\pm L$. Consequently, the symmetries of the problem impose the following relations:

$$\begin{aligned} f(y, v_x, v_y, v_z \mid F) &= f(y, v_x, v_y, -v_z \mid F)\\ &= f(-y, v_x, -v_y, v_z \mid F)\\ &= f(y, -v_x, v_y, v_z \mid -F) \end{aligned} \tag{10}$$

where the notation $f(y, \mathbf{v} \mid F)$ has been used to denote the distribution function at a given value F of the external force. Thus, $n(y \mid F)=n(-y \mid F)=n(y \mid -F)$, $u_x(y \mid F)=u_x(-y \mid F)=-u_x(y \mid -F)$, $u_y=u_z=0$, and $T(y \mid F)=T(-y \mid F)=T(y \mid -F)$.

Henceforth, we specialize to Maxwell molecules, i.e., $\nu \propto p/T$, although the results could be easily extended to other interactions. In order to simplify the expressions it is useful to introduce dimensionless quantities

$$\mathbf{v}^*=(k_{\mathrm{B}}T_0/m)^{-1/2}\,\mathbf{v} \tag{11}$$

$$s(y)=(k_{\mathrm{B}}T_0/m)^{-1/2}\int_0^y dy'\,\nu(y') \tag{12}$$

$$f^*(s, \mathbf{v}^*) = n_0^{-1}(k_B T_0/m)^{3/2} f(y, \mathbf{v}) \tag{13}$$

$$p^* = p_0^{-1} p \tag{14}$$

$$F^* = m^{-1}(k_B T_0/m)^{-1/2} \nu_0^{-1} F \tag{15}$$

where the subscript 0 refers to quantities at $y = 0$. As a consequence, Eq. (9) can be rewritten as

$$v_y^* \frac{\partial f^*}{\partial s} + F^* \frac{T^*}{p^*} \frac{\partial f^*}{\partial v_x^*} = -f^* + f_{\mathrm{LE}}^* \tag{16}$$

The scaled space variable s measures distance in units of a local mean free path. Thus, the value $s = \varepsilon^{-1}$ at the boundary $y = L$ represents an inverse Knudsen number. The force unit chosen in Eq. (15) produces a velocity increment of the order of the thermal velocity over a mean free time. The scaled force F^* represents the uniformity parameter of the problem and measures the departure from equilibrium. The two independent parameters ε and F^* characterize the nonequilibrium stationary state.

In the following it is understood that only the dimensionless quantities occur, unless otherwise specified, and the asterisks will be deleted. From Eq. (16) one can get the balance equations:

$$\frac{\partial P_{yy}}{\partial s} = 0 \tag{17}$$

$$\frac{\partial P_{xy}}{\partial s} = F \tag{18}$$

$$\frac{\partial q_y}{\partial s} + P_{xy} \frac{\partial u_x}{\partial s} = 0 \tag{19}$$

Equation (18) shows that $P_{xy} = Fs$ exactly. If the velocity profile is known, Eq. (19) can be used to obtain q_y.

The formal solution of Eq. (16) is

$$f(s, \mathbf{v}) = \frac{1}{v_y} \int_{\varepsilon^{-1}}^{s} ds_1 \exp\left\{ -\frac{1}{v_y} \left[s - s_1 + \int_{s_1}^{s} ds_2 \frac{T(s_2)}{p(s_2)} F \frac{\partial}{\partial v_x} \right] \right\} f_{\mathrm{LE}}(s_1, \mathbf{v})$$
$$+ \exp\left[-\frac{1}{v_y} \int_{\varepsilon^{-1}}^{s} ds_1 \frac{T(s_1)}{p(s_1)} F \frac{\partial}{\partial v_x} \right] f_b(\mathbf{v}) \tag{20}$$

where $f_b(\mathbf{v})$ is the boundary condition at $s = \varepsilon^{-1}$. In this paper, we are interested in the properties of the gas in the bulk rather than near the boundaries. Therefore, we will take the limit of zero Knudsen number

$(\varepsilon \to 0)$ and no specific boundary conditions will be imposed. Instead, we will look for a "normal" solution of Eq. (9), i.e., a solution that depends on y entirely through a functional dependence on n, $\mathbf{u}$, and T. Consequently, it is preferable to rewrite the formal solution of Eq. (16) in the form given by Eq. (8) with

$$\begin{aligned} \mathscr{L} &= \left(v_y \partial_s + F \frac{T}{p} \mathscr{D}_x \right)^{-1} - 1 \\ &= \sum_{k=1}^{\infty} (-1)^k \left(v_y \partial_s + F \frac{T}{p} \mathscr{D}_x \right)^k \end{aligned} \tag{21}$$

where we have introduced the shorthand notations $\partial_s \equiv \partial/\partial s$ and $\mathscr{D}_x \equiv \partial/\partial v_x$. The consistency conditions for determining the hydrodynamic fields are

$$\int d\mathbf{v}\, \mathscr{L} f_{\text{LE}} = 0 \tag{22}$$

$$\int d\mathbf{v}\, v_x \mathscr{L} f_{\text{LE}} = 0 \tag{23}$$

$$\int d\mathbf{v}\, v^2 \mathscr{L} f_{\text{LE}} = 0 \tag{24}$$

It is important to note that the operator ∂_s does not commute with T/p. Consequently,

$$\begin{aligned} \left(v_y \partial_s + F \frac{T}{p} \mathscr{D}_x \right)^k &= v_y^k \partial_s^k + F v_y^{k-1} \mathscr{D}_x \sum_{l=0}^{k-1} \partial_s^l \frac{T}{p} \partial_s^{k-1-l} \\ &\quad + F^2 v_y^{k-2} \mathscr{D}_x^2 \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \partial_s^l \frac{T}{p} \partial_s^r \frac{T}{p} \partial_s^{k-2-l-r} \\ &\quad + \cdots \end{aligned} \tag{25}$$

where the ellipsis represents terms of third and higher order in $\mathscr{D}_x$, which give vanishing contributions in Eqs. (22)–(24). Insertion of Eq. (21) into Eqs. (22)–(24) yields, respectively,

$$0 = \sum_{k=1}^{\infty} (2k-1)!!\, \partial_s^{2k} p T^{k-1} \tag{26}$$

$$\begin{aligned} 0 &= \sum_{k=1}^{\infty} (2k-1)!!\, \partial_s^{2k} u_x p T^{k-1} \\ &\quad + F \left[1 + \sum_{k=2}^{\infty} (2k-3)!! \sum_{l=0}^{2k-2} \partial_s^l \frac{T}{p} \partial_s^{2k-2-l} p T^{k-2} \right] \end{aligned} \tag{27}$$

$$0 = \sum_{k=1}^{\infty} (2k-1)!!\, \partial_s^{2k}[(2k+3)\,T + u_x^2]\; pT^{k-1}$$
$$+ 2F\left[u_x + \sum_{k=2}^{\infty} (2k-3)!! \sum_{l=0}^{2k-2} \partial_s^l \frac{T}{p}\, \partial_s^{2k-2-l} u_x\, pT^{k-2}\right]$$
$$+ 2F^2\left[\frac{T}{p} + \sum_{k=2}^{\infty} (2k-3)!! \sum_{l=0}^{2k-2} \sum_{r=0}^{2k-2-l} \partial_s^l \frac{T}{p}\, \partial_s^r \frac{T}{p}\, \partial_s^{2k-2-l-r} pT^{k-2}\right] \tag{28}$$

The steps leading to these equations are worked out in ref. 6 and will not be repeated here. Equations (26)–(28) are the conditions that the fields $p(s\,|\,F)$, $T(s\,|\,F)$, and $u_x(s\,|\,F)$ must satisfy for arbitrary values of F. They are really equivalent to integral equations [see Eq. (20)] and this is why derivatives of all orders are present. Nevertheless, this representation will be useful for evaluating the fields in powers of F, as the next section shows.

3. HYDRODYNAMIC FIELDS

In order to solve the set of equations (26)–(28), we perform a perturbation expansion in powers of F:

$$u_x(s\,|\,F) = u^{(1)}(s)\,F + u^{(3)}(s)\,F^3 + u^{(5)}(s)\,F^5 + \mathcal{O}(F^7) \tag{29}$$

$$p(s\,|\,F) = 1 + p^{(2)}(s)\,F^2 + p^{(4)}(s)\,F^4 + \mathcal{O}(F^6) \tag{30}$$

$$T(s\,|\,F) = 1 + T^{(2)}(s)\,F^2 + T^{(4)}(s)\,F^4 + \mathcal{O}(F^6) \tag{31}$$

Because of our choice of units, $p^{(2k)}(0) = T^{(2k)}(0) = 0$. We also choose a reference frame stationary with the fluid at $s = 0$, so that $u^{(2k-1)}(0) = 0$. Substitution of Eqs. (29)–(31) into Eqs. (26)–(28) allows one to get the expansion coefficients recursively. In this section we are going to obtain the coefficients up to fifth order, i.e., those explicitly written in Eqs. (29)–(31).

To first order in F, Eq. (27) yields

$$0 = \sum_{k=1}^{\infty} (2k-1)!!\, \partial_s^{2k} u^{(1)} + 1 \tag{32}$$

whose solution is

$$u^{(1)}(s) = -\tfrac{1}{2}s^2 \tag{33}$$

This is the parabolic velocity profile that is characteristic of the Poiseuille flow.

Next, to second order in F, Eqs. (26) and (28) become, respectively,

$$0=\sum_{k=1}^{\infty}(2k-1)!!\,\partial_s^{2k}[p^{(2)}+(k-1)\,T^{(2)}] \tag{34}$$

$$\begin{aligned}0=&\sum_{k=1}^{\infty}(2k-1)!!\,\partial_s^{2k}\{(2k+3)[p^{(2)}+kT^{(2)}]+(u^{(1)})^2\}\\&+2\sum_{k=2}^{\infty}(2k-1)!!\,\partial_s^{2k-2}u^{(1)}+2u^{(1)}+2\end{aligned} \tag{35}$$

Equations (34) and (35) show that both $p^{(2)}(s)$ and $T^{(2)}(s)$ are polynomials of degree less than or equal to 4. The coefficients are obtained by substitution and the result is

$$p^{(2)}(s)=\tfrac{6}{5}s^2 \tag{36}$$

$$T^{(2)}(s)=\tfrac{19}{25}s^2-\tfrac{1}{30}s^4 \tag{37}$$

Proceeding in a similar way, and after rather tedious calculations, we get

$$u^{(3)}(s)=-\frac{2737}{25}s^2-\frac{13}{15}s^4 \tag{38}$$

$$p^{(4)}(s)=-\frac{326,622}{125}s^2-\frac{223}{25}s^4 \tag{39}$$

$$T^{(4)}(s)=-\frac{25,472,912}{3125}s^2-\frac{33,157}{750}s^4-\frac{43}{250}s^6 \tag{40}$$

$$u^{(5)}(s)=\frac{11,294,974,736}{3125}s^2+\frac{5,168,033}{375}s^4+\frac{5914}{375}s^6 \tag{41}$$

By following the same recursive procedure, one can get higher-order coefficients, although the algebra becomes progressively more tedious. Inspection of Eqs. (26)–(28) shows that $p^{(k)}$, $u^{(k)}$, and $T^{(k)}$ are (even) polynomials in s of degrees k, $k+1$, and $k+2$, respectively:

$$p^{(k)}(s)=\sum_{l=1}^{k/2}p_{2l}^{(k)}s^{2l} \tag{42}$$

$$u^{(k)}(s)=\sum_{l=1}^{(k+1)/2}u_{2l}^{(k)}s^{2l} \tag{43}$$

$$T^{(k)}(s)=\sum_{l=1}^{(k+2)/2}T_{2l}^{(k)}s^{2l} \tag{44}$$

Consequently, the expansions (29)–(31) in powers of F can be recast into expansions in powers of s:

$$u_x(s\,|\,F)=s^2Fu_2(F)+s^4F^3u_4(F)+s^6F^5u_6(F)+\mathcal{O}(s^8) \tag{45}$$

$$p(s\,|\,F)=1+s^2F^2p_2(F)+s^4F^4p_4(F)+\mathcal{O}(s^6) \tag{46}$$

$$T(s\,|\,F)=1+s^2T_2(F)+s^4F^2T_4(F)+s^6F^4T_6(F)+\mathcal{O}(s^8) \tag{47}$$

where

$$u_{2l}(F)=\sum_{k=0}^{\infty} u_{2l}^{(l-1+2k)}F^{2k} \tag{48}$$

$$p_{2l}(F)=\sum_{k=0}^{\infty} p_{2l}^{(l+2k)}F^{2k} \tag{49}$$

$$T_{2l}(F)=\sum_{k=0}^{\infty} T_{2l}^{(l-2+2k)}F^{2k} \tag{50}$$

From Eqs. (33) and (36)–(41) one gets

$$u_2(F)=-\frac{1}{2}-\frac{2737}{25}F^2+\frac{11{,}294{,}974{,}736}{3125}F^4+\mathcal{O}(F^6) \tag{51}$$

$$u_4(F)=-\frac{13}{15}\left[1-\frac{397{,}541}{25}F^2+\mathcal{O}(F^4)\right] \tag{52}$$

$$u_6(F)=\frac{5914}{375}+\mathcal{O}(F^2) \tag{53}$$

$$p_2(F)=\frac{6}{5}\left[1-\frac{54{,}437}{25}F^2+\mathcal{O}(F^4)\right] \tag{54}$$

$$p_4(F)=-\frac{223}{25}+\mathcal{O}(F^2) \tag{55}$$

$$T_2(F)=F^2\left[\frac{19}{25}-\frac{25{,}472{,}912}{3125}F^2+\mathcal{O}(F^4)\right] \tag{56}$$

$$T_4(F)=-\frac{1}{30}\left[1+\frac{33{,}157}{25}F^2+\mathcal{O}(F^4)\right] \tag{57}$$

$$T_6(F)=-\frac{43}{250}+\mathcal{O}(F^2) \tag{58}$$

These equations clearly indicate that the series expansions in powers of the external force are only asymptotic. Thus, their practical usefulness is restricted to small forces. On the other hand, the expansions give information about how the hydrodynamic profiles are expected to be affected by a modification of the force. This information is complemented by the exact solution of Eq. (16) found in ref. 6 for the special value $F \simeq 2.52$. For that value, $u_2 \simeq -146$, $T_2 \simeq -164$, $u_4 = u_6 = \cdots = 0$, $p_2 = p_4 = \cdots = 0$, and $T_4 = T_6 = \cdots = 0$. Comparison with Eqs. (51)–(58) indicates that $u_2(F)$, $-u_4(F)$, $p_2(F)$, and $T_2(F)$ possibly decrease monotonically, while the behavior of $T_4(F)$ cannot be monotonic. The fact that all the coefficients except u_2 and T_2 vanish in the special solution suggests that a hidden relationship between those coefficients might exist. Nevertheless, our results are not sufficiently extensive to elucidate such a relationship.

The results of this section allow us to obtain explicitly the connection between the space variables y and s up to fifth order in F. From Eq. (12) one has

$$\begin{aligned} y &= \int_0^s ds' \, \frac{T(s')}{p(s')} \\ &= s - F^2 s^3 \left(\frac{11}{25} + \frac{1}{30} s^2 \right) \\ &\quad - F^4 s^3 \left(\frac{17,307,362}{3125} + \frac{26,071}{750} s^2 + \frac{33}{250} s^4 \right) + \mathcal{O}(F^6) \end{aligned} \tag{59}$$

4. VELOCITY DISTRIBUTION FUNCTION AND VELOCITY MOMENTS

Once the hydrodynamic fields are known in the form of series in powers of F, the velocity distribution function f can be explicitly derived from Eqs. (8) and (21). First, we write

$$f = f_{\mathrm{LE}}(1 + \varphi) \tag{60}$$

Then we perform an expansion of φ in powers of F in the same spirit as in the expansions (29)–(31):

$$\varphi(s, \mathbf{V} \mid F) = \varphi^{(1)}(s, \mathbf{V}) \, F + \varphi^{(2)}(s, \mathbf{V}) \, F^2 + \varphi^{(3)}(s, \mathbf{V}) \, F^3 + \mathcal{O}(F^4) \tag{61}$$

The coefficients can be obtained recursively. For the sake of conciseness, we omit here the technical details and just quote the expressions for the three first coefficients. They are

$$\varphi^{(1)} = V_x[1 + V_y(s - V_y)] \tag{62}$$

$$\begin{aligned}\varphi^{(2)} = {}& \tfrac{1}{15}(V^2 - 5)\, V_y s^3 + \tfrac{1}{5}(5V_x^2 - V^2)V_y^2 s^2 \\ & + \tfrac{1}{25}\{(10V_y^2 - 19)\, V^2 - 5[5V_x^2(3V_y^2 - 2) - 5V_y^2 + 3]\}\, V_y s \\ & + \tfrac{1}{25}\{5[5V_x^2(3V_y^4 - 3V_y^2 + 1) - 5V_y^4 + 8V_y^2 - 5] \\ & - V^2V_y^2(10V_y^2 - 19)\}\end{aligned} \tag{63}$$

$$\begin{aligned}\varphi^{(3)} = {}& V_x\{\tfrac{1}{30}V_y s^5 + \tfrac{1}{30}(4V^2 - 29)\, V_y^2 s^4 - \tfrac{1}{75}[(6V_y^2 - 1)\, 10V^2 - 75V_x^2V_y^2 \\ & - 13(15V_y^2 + 11)]\, V_y s^3 + \tfrac{1}{25}[(70V_y^2 - 53)\, V_y^2V^2 - (2V_y^2 - 1)\, 75V_x^2V_y^2 \\ & - 40V_y^4 - 225V_y^2 - 30]\, s^2 - \tfrac{1}{25}[2(75V_y^4 - 77V_y^2 + 19)\, V^2 \\ & - 75(5V_y^4 - 4V_y^2 + 1)\, V_x^2 + 75V_y^4 - 722V_y^2 - 5417]\, V_y s \\ & + \tfrac{1}{25}[(150V_y^4 - 164V_y^2 + 57)\, V_y^2V^2 - 25(15V_y^6 - 15V_y^4 + 6V_y^2 - 1)\, V_x^2 \\ & + 75V_y^6 - 877V_y^4 - 5265V_y^2 - 75]\}\end{aligned} \tag{64}$$

We could take advantage of our knowledge of u_x, p, and T up to fifth order in F to evaluate φ up to that order as well, but the expressions of $\varphi^{(4)}$ and $\varphi^{(5)}$ are excessively tedious and lengthy. On the other hand, the general structure of $\varphi^{(k)}$ can be induced from Eqs. (62)–(64):

$$\varphi^{(k)}(s, \mathbf{V}) = \sum_{l=0}^{2k-1} \phi_l^{(k)}(\mathbf{V})\, s^l \tag{65}$$

where $\phi_l^{(k)}(\mathbf{V})$ is a polynomial of degree $3k - l$ in $\mathbf{V}$ and of parity l with respect to V_y and parity k with respect to V_x, on account of Eq. (10).

We define the velocity moments of the distribution as

$$M_{\alpha,\beta,\gamma}(s \mid F) = \int d\mathbf{V}\; V_x^\alpha V_y^\beta V_z^\gamma f(s, \mathbf{V} \mid F) \tag{66}$$

with $\alpha, \beta, \gamma = 0, 1, 2, \ldots$. Insertion of Eq. (60) into Eqs. (66) gives the decomposition

$$M_{\alpha,\beta,\gamma}(s \mid F) = M_{\alpha,\beta,\gamma}^{\mathrm{LE}}(s \mid F) + \Delta M_{\alpha,\beta,\gamma}(s \mid F) \tag{67}$$

where

$$\begin{aligned}M_{\alpha,\beta,\gamma}^{\mathrm{LE}}(s \mid F) &= \int d\mathbf{V}\; V_x^\alpha V_y^\beta V_z^\gamma f_{\mathrm{LE}}(s, \mathbf{V} \mid F) \\ &= C_\alpha C_\beta C_\gamma\, p(s \mid F)[T(s \mid F)]^{(\alpha + \beta + \gamma - 2)/2}\end{aligned} \tag{68}$$

and

$$\Delta M_{\alpha,\beta,\gamma}(s\,|\,F)=\int d\mathbf{V}\; V_x^{\alpha} V_y^{\beta} V_z^{\gamma} f_{\mathrm{LE}}(s,\mathbf{V}\,|\,F)\,\varphi(s,\mathbf{V}\,|\,F) \tag{69}$$

In Eq. (68), $C_0=1$, $C_\alpha=(\alpha-1)!!$ if $\alpha=2, 4, 6,\ldots$, and $C_\alpha=0$ if $\alpha=1, 3, 5,\ldots$. The kinetic equation (16) generates the following hierarchy for the moments:

$$\frac{\partial}{\partial s} M_{\alpha,\beta+1,\gamma}+\alpha\frac{\partial u_x}{\partial s} M_{\alpha-1,\beta+1,\gamma}-\alpha F\frac{T}{p} M_{\alpha-1,\beta,\gamma}=-\Delta M_{\alpha,\beta,\gamma} \tag{70}$$

This hierarchy cannot be solved for arbitrary F, as moments of a given degree are coupled to moments of higher degree. However, the knowledge of φ up to third order in F allows one to get $\Delta M_{\alpha,\beta,\gamma}$ up to that order:

$$\Delta M_{\alpha,\beta,\gamma}(s\,|\,F)=M^{(1)}_{\alpha,\beta,\gamma}(s)\,F+M^{(2)}_{\alpha,\beta,\gamma}F^2+M^{(3)}_{\alpha,\beta,\gamma}F^3+\mathcal{O}(F^4) \tag{71}$$

By using Eqs. (62)–(64), we get

$$M^{(1)}_{\alpha,\beta,\gamma}(s)=C_{\alpha+1}C_\gamma(C_{\beta+1}s-\beta C_\beta) \tag{72}$$

$$\begin{aligned}M^{(2)}_{\alpha,\beta,\gamma}(s)={}&C_\alpha C_\gamma\{\tfrac{1}{15}C_{\beta+1}(\alpha+\beta+\gamma-1)\,s^3+\tfrac{1}{5}C_{\beta+2}(4\alpha-\beta-\gamma)\,s^2\\&-\tfrac{1}{25}C_{\beta+1}[-10\beta^2+\beta(65\alpha-10\gamma-11)+99\alpha-\gamma+21]\,s\\&+\tfrac{1}{25}C_\beta[-10\beta^3+\beta^2(65\alpha-10\gamma-41)+\beta(204\alpha-21\gamma-31)\\&+164\alpha-11\gamma]\}\end{aligned} \tag{73}$$

$$\begin{aligned}M^{(3)}_{\alpha,\beta,\gamma}(s)={}&C_{\alpha+1}C_\gamma\{-\tfrac{1}{60}C_{\beta+1}(\alpha+\beta+\gamma-2)\,s^5\\&+\tfrac{1}{60}C_\beta[9\beta^2+\beta(9\alpha+9\gamma-1)+8(\alpha+\gamma-1)]\,s^4\\&+\tfrac{1}{150}C_{\beta+1}[-120\beta^2+\beta(30\alpha-120\gamma-313)\\&+137\alpha-163\gamma+266]\,s^3\\&-\tfrac{1}{50}C_\beta[-140\beta^3+5\beta^2(32\alpha-28\gamma-175)\\&+\beta(615\alpha-435\gamma-1209)+2(218\alpha-157\gamma-218)]\,s^2\\&+\tfrac{1}{25}C_{\beta+1}[-150\beta^3+\beta^2(225\alpha-150\gamma-1421)\\&+2\beta(602\alpha-373\gamma-1890)+1545\alpha-930\gamma+2377]\,s\\&-\tfrac{1}{25}C_\beta[-150\beta^4+\beta^3(225\alpha-150\gamma-2011)\\&+\beta^2(1814\alpha-1186\gamma-8837)+\beta(4424\alpha-2851\gamma-9811)\\&+5(562\alpha-363\gamma-562)]\}\end{aligned} \tag{74}$$

In general, $M^{(k)}_{\alpha,\beta,\gamma}(s)$ is a polynomial of parity β and degree $2k-1$ (if $\beta=\text{odd}$) or $2k-2$ (if $\beta=\text{even}$). Besides, $M^{(k)}_{\alpha,\beta,\gamma}(s)=0$ if α and k have a different parity. In particular, the pressure tensor and the heat flux vector are

$$P_{xx}(s\,|\,F)=1+\left(\frac{328}{25}+\frac{14}{5}s^2\right)F^2+\mathcal{O}(F^4) \tag{75}$$

$$P_{yy}(s\,|\,F)=1-\frac{306}{25}F^2+\mathcal{O}(F^4) \tag{76}$$

$$P_{zz}(s\,|\,F)=1+\left(-\frac{22}{25}+\frac{4}{5}s^2\right)F^2+\mathcal{O}(F^4) \tag{77}$$

$$P_{xy}(s\,|\,F)=sF+\mathcal{O}(F^5) \tag{78}$$

$$q_x(s\,|\,F)=-F\left[1-\left(\frac{21{,}162}{25}+\frac{159}{5}s^2+\frac{29}{30}s^4\right)F^2+\mathcal{O}(F^4)\right] \tag{79}$$

$$q_y(s\,|\,F)=\frac{1}{3}s^3F^2+\mathcal{O}(F^4) \tag{80}$$

Equations (76), (78), and (80) are consistent with the balance equations (17)–(19). In fact, the result $P_{xy}=sF$ is exact to any order in F. We could use Eq. (19) and the knowledge of u_x to fifth order to get q_y up to seventh order. Equation (79) shows that the force induces heat transport along the force direction, even in the absence of gradients along that direction.

According to Eqs. (75)–(77), the normal stresses created in the system by the external force are such that $P_{xx}>p>P_{zz}>P_{yy}$. Also, $-q_x/F$ and $q_y/(\frac{1}{3}s^3F^2)$ tend to decrease and increase, respectively, when F increases. These qualitative behaviors agree with the results obtained from the exact solution for $F\simeq 2.52$, in which case[6] $P_{xx}\simeq 2.70$, $p=1$, $P_{zz}\simeq 0.22$, $P_{yy}\simeq 0.08$, and $q_y/(\frac{1}{3}s^3F^2)=-2u_2\simeq 291$.

5. DISCUSSION

In this paper we have been concerned with the stationary flow of a dilute gas confined between two infinite parallel plates and subject to the action of a constant external force $\mathbf{F}=F\hat{\mathbf{x}}$ parallel to the plates. The force drives the system out of equilibrium and induces hydrodynamic gradients along the direction y normal to the plates. In the limit of small force strength, the state of the gas mimics the Poiseuille flow due to a small pressure difference along the x direction. In both cases, the density and temperature are constant, the velocity profile is quadratic, and the pressure tensor is given by Newton's law. Beyond this linear regime, the response of the system is more complicated and a much richer rheological state arises.

In order to get explicit results, we have used the Bhatnagar–Gross–Krook (BGK) kinetic model of the Boltzmann equation. Also, we have restricted ourselves to Maxwell molecules. The BGK equation for this problem has been solved in the bulk by means of a perturbation expansion in powers of the force. More specifically, the profiles of pressure, temperature, and velocity have been obtained to fifth order in F, and the velocity distribution function has been derived to third order, the expansion coefficients being polynomials in the space variable. The results clearly indicate that the expansion is asymptotic, but not convergent. This means that the state of the system is singular at equilibrium, i.e., at $F=0$. This is not surprising, as similar asymptotic series appear, for instance, in the stationary Fourier and Couette flows$^{(10)}$ and in the uniform shear flow.$^{(11)}$

The perturbation solution obtained here complements the closed exact solution obtained previously$^{(6)}$ for a special value of the force. In both cases, for instance, the normal stresses follow the behavior $P_{xx}-p>0>P_{zz}-p>P_{yy}-p$. It is interesting to note that the profiles in the special solution are much simpler than in the general solution. This suggests that a certain relationship among the expansion coefficients could exist.

Let us compare the profiles obtained in this paper with the Navier–Stokes predictions. In the units chosen in this paper, the Navier–Stokes constitutive equations are

$$P_{xx}^{\mathrm{NS}}=P_{yy}^{\mathrm{NS}}=P_{zz}^{\mathrm{NS}} \tag{81}$$

$$P_{xy}^{\mathrm{NS}}=-\frac{\partial u_x}{\partial s} \tag{82}$$

$$q_x^{\mathrm{NS}}=0 \tag{83}$$

$$q_y^{\mathrm{NS}}=-\frac{5}{2}\frac{\partial T}{\partial s} \tag{84}$$

In Eqs. (82) and (84) we have taken into account that the shear viscosity and the thermal conductivity in the BGK model are $\eta=p/\nu$ and $\kappa=\frac{5}{2}k_{\mathrm{B}}\,p/m\nu$, respectively. Insertion into the exact balance equations (17)–(19) yields the hydrodynamic profiles:

$$p^{\mathrm{NS}}=1 \tag{85}$$

$$u_x^{\mathrm{NS}}=-\tfrac{1}{2}Fs^2 \tag{86}$$

$$T^{\mathrm{NS}}=1-\tfrac{1}{30}F^2s^4 \tag{87}$$

Equations (85) and (86) are exact to first and third order, respectively, in F. Nevertheless, comparison between Eqs. (37) and (87) shows a discrepancy in the temperature profile to second order in the force. This discrepancy is due to the different scaling assumed in each approach. In this work we have assumed first that the Knudsen number vanishes ($\varepsilon \to 0$) and then that the external force F (relative to the one producing a finite change of velocity over a mean free time) is weak. In the Navier–Stokes equations, on the other hand, one assumes $F \sim \varepsilon^2$.(7) To clarify this point, let us introduce the variables $\tilde{s} = \varepsilon s$ and $\tilde{F} = \varepsilon^{-2} F$. The variable $\tilde{s}$ measures the distance in units of the system size, so that $\tilde{s} = \pm 1$ at the boundaries. The parameter $\tilde{F}$ measures the external force relative to the one producing a finite change of velocity over times in which the effect of the viscosity is felt. In terms of these quantities, the temperature profile derived in this paper becomes

$$T = 1 + \left(\tfrac{19}{25}\tilde{s}^2\varepsilon^2 - \tfrac{1}{30}\tilde{s}^4\right)\tilde{F}^2 + \mathcal{O}(\varepsilon^2) \qquad (88)$$

In the limit $\varepsilon \to 0$, Eq. (88) agrees with Eq. (87).

The method followed here to get the hydrodynamic profiles cannot be directly applied to the Boltzmann equation, even for Maxwell molecules. Instead, one has to deal with a moment hierarchy similar to Eq. (70). However, if one assumes that the expansion coefficients of the moments have the same polynomial structure as found here, the hierarchy can be solved recursively. Working along these lines, we plan to obtain the first few terms in the perturbation expansions of the hydrodynamic fields and the fluxes corresponding to the Boltzmann equation for Maxwell molecules.

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